“A Reduced-Dimension, Static, Linearized Kalman Filter and Smoother”

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OVERVIEW

An approximate Kalman filter and smoother, based on approximations of the state estimation error covariance matrix, will be described. Approximations include. a Auction of the effective state dimension, use of a static asymptotic error limit, and a time-invariant linearization of the dynamic model for error integration. The approximations lead to dramatic computational savings in applying estimation theory to large, complex systems. Examples of oceanographic applications will be presented analyzing altimeter data from TOPEX/POSEIDON, an ongoing joint U.S.-French oceanographic satellite mission.

THE PROBLEM

Practical applications of Kalman filtering are hampered by the large computational requirements involved in the time-integration of the estimation error covariance matrix (Riccati equation). The error is a square matrix of model dimension and evolves according to model dynamics, which requires the size-of-the-model times more computational resources than a straight model simulation without assimilation. Therefore, direct applications of Kalman filtering to state-of-the-art models will always remain unfeasible. Yet various approximations to the filter that take advantage of certain aspects of the system can be effective. The difficulty of error integration stems from the size and continuous time integration of the error matrix, for which approximations will be described in turn,

STATE DIMENSION

The grid size of a model and the resulting model dimension is often dictated by numerical accuracy and stability. On the other hand, most energetic scales are typically much larger than the smallest grid spacing, and available observations are often sparse. Then, extraction and assimilation of the measurements’ large-scale information may be most effective in terms of the amount of improvements made in the estimate for the amount of computations involved. The idea would be to approximate the model error covariance with one with fewer degrees of freedom (Fukumori and Malanotte-Rizzoli, 1994).

For example, suppose there exists an approximation, x’, with a smaller dimension than the original model state (x),

$$x(t) - \bar{\mathbf{x}} \approx B\, x'(t) \qquad (1)$$

The approximation is defined, without loss of generality, around some prescribed state, $\bar{\mathbf{x}}$. Matrix B is a transformation that defines x’, Then, the error covariance of x (P) may be approximated by the error of $\mathbf{x}'$ ($\mathbf{P}'$) by,

$$P(t) \approx \mathbf{B P'(t) B}^T \qquad (2)$$

which can be substituted into the Kalman gain for the original model of x. Owing to the smaller dimension, derivation of $\mathbf{P}'$ would be much easier than a direct computation of P.

The equation (model) for x’, from which P’ is computed, maybe obtained by simply combining B with the model for x. Denote the original dynamic model by a vector function F, which describes the time evolution of model state, x(t);

$$\mathbf{x}(t+1) = \mathbf{F}(\mathbf{x}(t)) \qquad (3)$$

Then, substituting eq (1) yields,

$$\mathbf{x}'(t+1) \approx B^* \mathbf{F}(\bar{\mathbf{x}}+\mathbf{B}\mathbf{x}'(t)) - B^* \bar{\mathbf{x}} \equiv F'(x'(t)) \qquad (4)$$

where B* is the pseudo inverse of B.

TIME INTEGRATION

When data are regularly assimilated, estimation errors often approach a steady-state limit. Using such limit throughout assimilation eliminates the need for the continuous error integration. Storage requirements for the smoother will be greatly reduced as well, as the time-varying error covariance need not be saved (Fukumori et al., 1993). For time-invariant linear systems, the Riccati equation can be shown under certain conditions to converge exponentially fast to a unique limit. In many situations, strict convergence does not occur, because of time varying models (including nonlinear ones) and/or aperiodic observations. However, experience shows that asymptotic errors, derived based on approximating such systems as time-invariant ones, can still be effective when used in Kahnan filtering and smoothing of time-varying systems.

Several efficient methods exist for deriving the asymptotic estimation error limit. One such method is the doubling algorithm (Anderson and Moore, 1979), which allows integration of the error in increasing time steps of powers of two. The doubling algorithm is a matrix recursion involving three matrices of state dimension, and, for reference, is given below;

$$\begin{aligned}\Phi(t+1) &= \Phi(t)\left[\mathbf{I} + \Psi(t)\Theta(t)\right]^{-1}\Phi(t)\\ \Psi(t+1) &= \Psi(t) + \Phi(t)\left[\mathbf{I} + \Psi(t)\Theta(t)\right]^{-1}\Psi(t)\Phi^{T}(t)\\ \Theta(t+1) &= \Theta(t) + \Phi^{T}(t)\Theta(t)\left[\mathbf{I} +- \Psi(t)\Theta(t)\right]^{-1}\Phi(t)\end{aligned} \tag{5}$$

The recursion is started from,

$$\Phi(1) = AT, \ V(l) = \mathbf{H}^{T}\mathbf{R}^{-1}\mathbf{H}, \Theta(1) = Q \tag{6}$$

where **A is** the state transition matrix and H is the observation matrix. Matrices R and Q are observation and process noise covariances, respectively. Matrix $\Theta(t)$ is the estimation error at time 2t.

NUMERICAL LINEARIZATION

For nonlinear models, the static filter may be derived using a time-invariant linearization of the model, and the corresponding state transition matrix, A‘, can be computed numerically for use in the doubling algorithm. For example, linearizing equation (4) around $\bar{\mathbf{x}}$,

$$\begin{aligned}\mathbf{x}'(t+1) &\approx B * \mathbf{F}(\bar{\mathbf{x}}+\mathbf{B}\mathbf{x}'(t)) - \mathbf{B}^{*}\bar{\mathbf{x}}\\ &\approx \mathbf{B}^{*}\mathbf{F}(\bar{\mathbf{x}}) + \mathbf{B}^{*}\frac{\partial \mathbf{F}}{ax} lI x'(t) - \mathbf{B}^{*}\bar{\mathbf{x}} \equiv \mathbf{B}^{*}\mathbf{F}(\bar{\mathbf{x}}) + \mathbf{A}'\mathbf{x}'(t) - \mathbf{B}^{*}\bar{\mathbf{x}}\end{aligned} \tag{7}$$

Then, each column of A‘, ai, may be obtained by,

$$\mathbf{a}_i = \mathbf{A}'\mathbf{e}_i \approx \mathbf{B}^{*}\mathbf{F}(\bar{\mathbf{x}}+\mathbf{B}\mathbf{e}_i) - \mathbf{B}^{*}\mathbf{F}(\bar{\mathbf{x}}) \tag{8}$$

where ei is the i ‘th column of the identity matrix, and the two terms on the right hand side can be evaluated numerically using the model. Other system matrices may be constructed easily likewise. Similar construction for linear models also facilitate derivation of approximate filters and smoothers for large, complex systems.

TOPEXPOSEIDON ANALYSIS

Results from an example of applying a static filter and smoother are shown in Figs 1 and 2. Fig 1 is a comparison of sea level anomalies along 12.5°N across the Pacific Ocean among measurements from TOPEX/POSEIDON, an assimilated estimate with an approximate smoother, and a numerical simulation without assimilation. The model is a linear reduced gravity shallow water model of the tropical Pacific Ocean, with parameters typical of the first baroclinic mode, and is forced by 12 hourly winds of the National Meteorological Center (NMC) analyses. The model has a 2’ zonal and 10 meridional resolution with a total state dimension exceeding 12,000 elements. An approximate filter and smoother was constructed using a 10° by 5“ grid and objective mapping as the transformation operator (B in eq 1). The coarse state has 831 elements and the assimilation required approximately 10 CPU minutes and 5 Mw of memory on a Cray YMP. In comparison, a direct application of a Kalman filter would have required 150 Mw of memory and several hundred CPU hours.

The simulation has qualitative similarities with the satellite measurements (Fig 1), but quantitatively accounts for only a few percent of the TOPEX observed sea level variance when averaged over the entire model domain. In comparison, the assimilated estimate resolves many of the features in the measurements, and accounts for nearly 78% of the observed sea level variance. The seasonal change of sea level

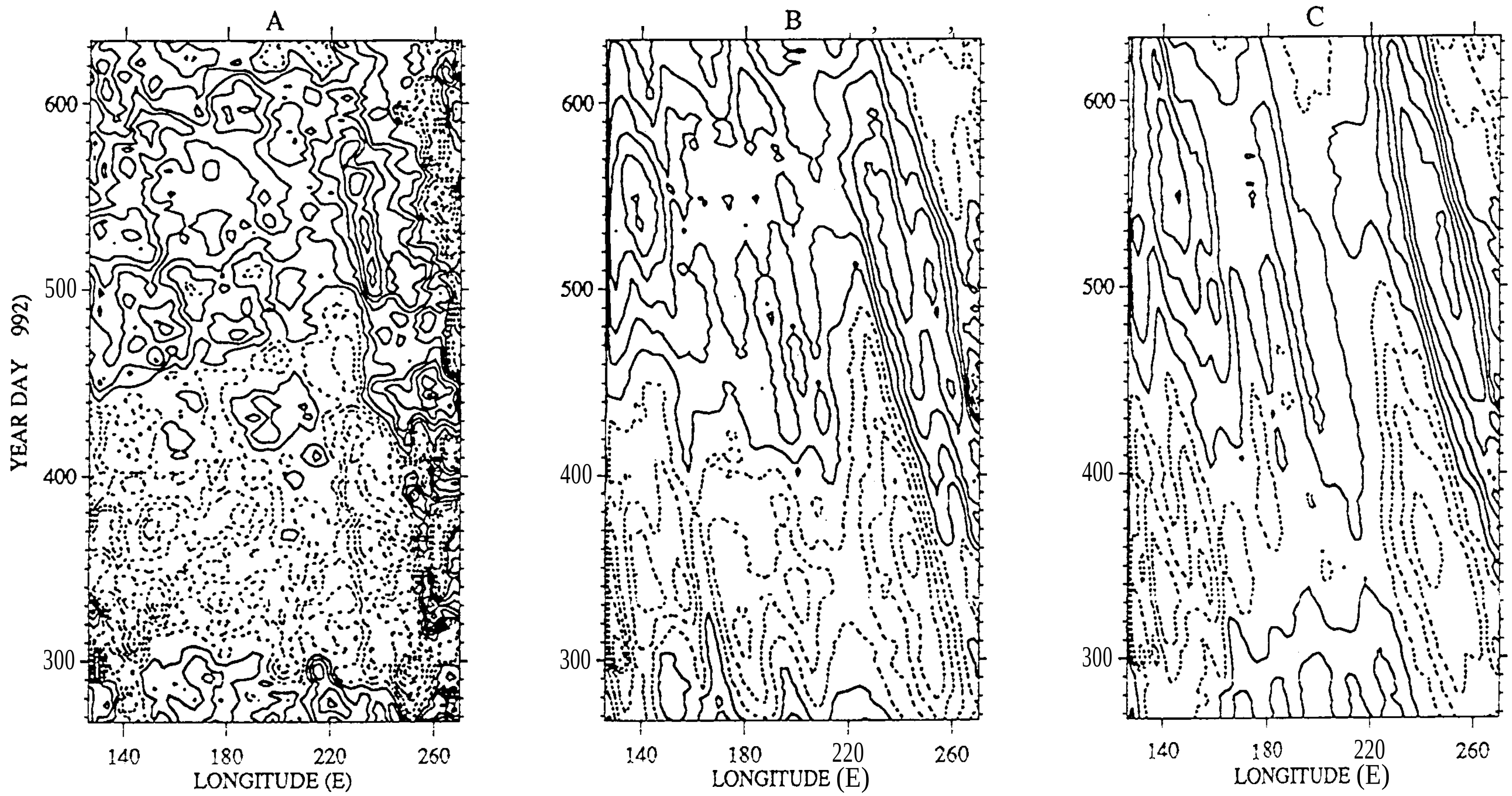

Fig 1. Longitude vs time plot of sea level anomalies along 12.5°N. The three figures are TOPEX/POSEIDON data (A), smoothed estimate (B), and model simulation (C), respectively. Contour interval is 3 cm. Dotted curves denote negative values. (A) was constructed by taking the raw data between 11.5"N and 13.5"N, and averaging them in three day and 2° zonal bins. The values were further smoothed for contouring purposes. Largest differences between (A) and (C) occur in the central Pacific.

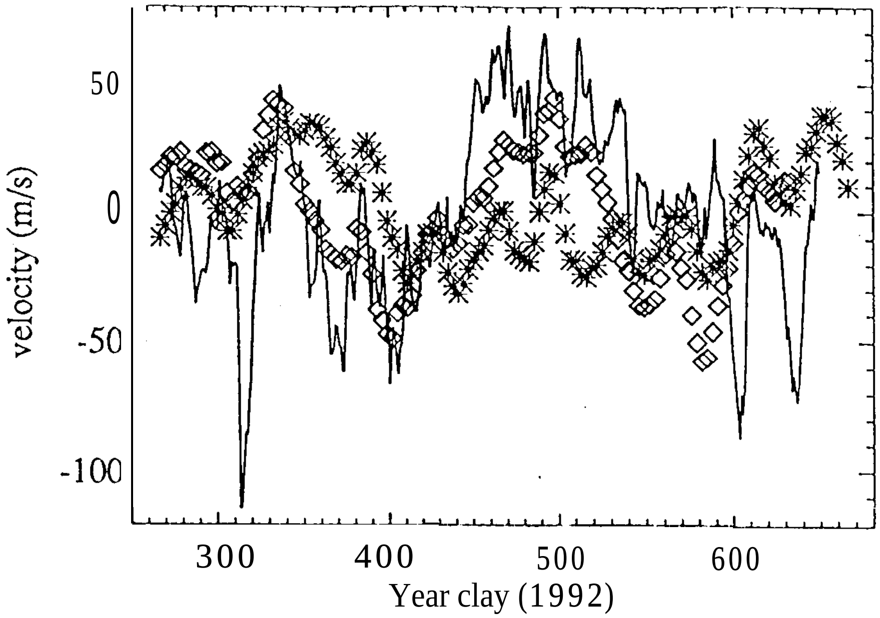

Fig 2. Zonal velocity anomalies (m/s) at 0°N, 140°W; TAO measurement at"25m (solid curve), smoothed estimate (diamond), model simulation (asterisk). Correlation coefficients with the data for simulation and assimilation are -4). 18 and 0.44, respectively.

at 12.5°N is associated with changes in the strength of the North Equatorial Countercurrent, and westward propagating signals are associated with Rossby waves.

Fig 2 shows comparisons between estimated zonal velocities and current meter observations from a Tropical Atmosphere and Ocean Array mooring (TAO; Hayes et al., 1991) at 140°W. Although assimilation was of only sea level data, current velocities of the assimilated result are closer to the observations than the model simulation is.

SUMMARY

Approximate Kalman filters and smoothers can be constructed that require less computational resources than otherwise and yet retain properties of the optimal solution to be effective. Examples of applying such approximations to models in estimating ocean circulation from TOPEX/POSEIDON altimeter data, such as the one above and others, will be discussed.

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